REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*
03/22/2002

2. REPORT DATE
Final

3. DATES COVERED *(From - To)*
14/11/1997 - 13/11/2001

4. TITLE AND SUBTITLE
Connectivity and Colorings of Graphs

5a. CONTRACT NUMBER
N00014-97-1-1067

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
C. A. Rodger
D. G. Hoffman
P. D. Johnson

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Auburn University
Auburn University, AL 36849

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Auburn University
Auburn University, AL 36849

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSORING/MONITORING AGENCY REPORT NUMBER

12. DISTRIBUTION AVAILABILITY STATEMENT
APPROVED FOR PUBLIC RELEASE

13. SUPPLEMENTARY NOTES

14. ABSTRACT
See attached page.

20020328 128

15. SUBJECT TERMS
network vulnerability, connectivity
list-colorings, analgamations, hamilton cycles,

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
					19b. TELEPONE NUMBER *(Include area code)*

Abstract

The research performed with the support of this grant has focussed on the graph theoretical issues of colorings of various forms, connectivity issues of a traditional form and as applied to network vulnerability, and some work on extremal and worst-case scenarios.

Coloring problems considered included the development of the method of amalgamations, which takes an "outline" version of a structure of interest and shows how to develop it into the completed version. This produced, for example, a constructive proof that there exist hamilton decompositions of complete multipartite graphs that are fair in the sense that the edges between pairs of parts are shared out evenly among the hamilton cycles. The method was also improved to the point where notable results concerning connected k-factorizations were obtained.

Several results were obtained on list-colorings, fractional colorings and greedy colorings of graphs. For example, a study of extensions of the classic theorem of Hall on systems of district representatives was applied to list colorings. Algorithmic methods were also developed for considering proper list-(multi)colorings.

Network vulnerability was an issue studied in connection with the Shields-Harary number of a graph. This involves enemies removing weighted vertices at a certain cost, trying to break the graph into components of small weight, while defenders arrange the weights on the vertices to exact the greatest payment possible by the enemy.

Detailed Final Report

I. Amalgamations, Hamilton Cycles and Connectivity

One of the successful lines of research associated with this grant has been the development of the powerful graph decomposition method, known as *amalgamations* of graphs. The idea is to "squash together" (formally, this is a graph homomorphism) vertices in some family of graphs, then to describe enough properties satisfied by the resulting structure that we can prove that *any* structure satisfying these properties can be "pulled apart" into a graph of the original type. Edge-colorings play a central role in the difficult step of "disentangling" the vertices.

To test the power of this method, we focussed on several hamilton decomposition problems, so connectivity was also a central issue in this study. We began by solving the problem: How big could a maximal set of edge-disjoint hamilton cycles in $G = K_{n,n}$ be? (Or, how many edge-disjoint hamilton cycles could one find in $K_{n,n}$ if a greedy algorithm was used?) The answer turns out to be anywhere from $n/4$ to $n/2$ [14]. We have since then tried to solve the corresponding problem when G is a complete multipartite graph, coming very close to a complete answer [39]. Amalgamations have worked very well, but in every case we have only been able to squash together vertices occurring in the same part. This makes it difficult to find the initial configurations; so further results on the technique are worth proving. Such improvements may also be necessary to solve the related problem of funding maximal sets of 1-factors in complete multipartite graphs.

One of the major steps forward in our knowledge of amalgamations was made using a result of Nash-Williams on laminar sets. It enabled us to solve the problem of deciding when an edge-colored multigraph with loops could be the amalgamation of an edge-colored complete graph in which each color class is connected [38]. This allowed us to address the captivating question: When can an edge-colored K_v be embedded as an edge-colored K_n in which each color class is a connected k-factor? (The color classes in the given K_v may be highly disconnected.) This requires disentangling the amalgamated graph without disconnecting *any* color class, a technique that is also discussed in [28]. Together, these methods enabled us to solve a problem that seems impossible to obtain using the "tried and tested" techniques; namely, to show when there exists a hamilton decomposition of $K_{n,n}$ once the edges of *any* specified 2-factor have been removed. We have since then extended the results to consider complete multipartite graphs [40].

Perhaps the most exciting result we found along these lines has the flavor of scheduling too. A hamilton decomposition of the complete multipartite graph is said to be *fair* if, between each pair of parts, the edges have been distributed as evenly as possible among the hamilton cycles. (For example, if each of 5 parts contains 10 vertices then each vertex has degree $(5-1)10 = 40$ so there are 20 hamilton cycles. Between each pair of parts there are $(10)(10) = 100$ edges; so each of the 20 hamilton cycles would have 5 edges between each pair of parts.) Amalgamations turned out to be the perfect tool that allowed us to solve this [32] when no other method seemed to make any headway.

Connectivity has also been of interest in studying two important families of graphs - cages and Johnson graphs. We have conjectured that cages, which are necessarily k-regular, are also k-connected. We did show that they are 3-connected [7], but making progress here seems to be very difficult. The defining properties of a cage (k-regular, fixed girth, as few vertices as possible) do not lend themselves well to this problem, though we still believe it to be true. It seems much more likely that it will be possible to show that they are all k-edge-connected. In [13] we found a new proof (from first principals) that showed that the Johnson graphs, which are also k-regular, do have connectivity k.

While dealing with partitions of the edges of the complete multipartite graph, each element of which induces a hamilton cycle, we wondered what would happen at the other extreme if small cycles were used instead of hamilton cycles. So several results were obtained for finding edge-disjoint 4-cycles in a complete multipartite graph, G. Unlike the case for hamilton decompositions, G now did not need to be regular. (This is also the case when considering maximal sets of edge-disjoint 1-factors in G, so resolvable versions of our results would help to solve this problem too.) We did manage to solve this problem, as well as consider what edges may be left out if no such decomposition exists. If $G = K_n$ then the possibility of omitting the edges in *any* specified 2-factor is settled in [27], and of omitting the edges in *any* specified forest is settled in [15]. If G is a complete multipartite graph then this general problem is addressed in terms of finding the least number of edges that occur in none of the cycles [25].

On a related problem, but with a different task, if 3-cycles are used instead of 4-cycles (and to some extend with 4-cycles too) then these results are of some statistical interest. We [33] have managed to extend known results of partitioning into 3-cycles the edges in $G = K_n - E(H)$ (K_n with the edges in a hamilton cycle removed); in particular G is replaced by a grid graph with a hamilton cycle in each row and in each column removed. Such a decomposition is of statistical interest when grounds are tested for, say, chemical remains in a dump site, and one can reasonably expect neighboring plots of land to have similar pollutants.

II. Graph Colorings

A. **List-colorings** [1, 4, 8, 10, 17, 21, 30, 31, 35].

[1] is a connection between the so-called "restricted choice numbers" of the complete bipartite graphs $K_{m,n}$, and the famous covering numbers $C(n,k,t)$, $n \geq k \geq t$, the size of a smallest collection of k-subsets of an n-set such that each t-subset of the n-set is a subset of some member of the collection. The connection between these important numbers and the attempt to color $K_{m,n}$ under certain restrictions is one that we hope to return to some day.

[35] is on a novel departure in list coloring, in which one assigns sets from a measure space to the vertices of a graph and attempts to properly color the vertices with subsets of the assigned sets of prescribed measure. We do not know where this might lead; we have other work on this matter in progress.

The other papers on list-colorings are about Hall's condition, a numerical condition, necessary for the existence of a proper list-(multi)coloring, discovered by Hilton and Johnson in 1989. The results are too numerous to summarize, but it is worth noting that [8] gives a partial solution of a major problem in this area, subsequently solved by Cropper, Gyarfas, and Lehel using some techniques from [8]. Also, [8] contains the beginnings of algorithmic methods and proofs in list-multicolorings. In [41], we give an algorithm for multi-list coloring a graph consisting of two cliques sharing a vertex. While such a network flow algorithm was independently discovered, ours is novel in that it uses algorithmic results on matroid intersections. Also, [10] and [30] are large contributions to the solution of another tough problem, since settled by Eslahchi and Matthew Johnson.

In [42], we observed that the notion of a multi-list coloring of graph can be extended to the much more general class of independence systems. There again, there is a corresponding necessary 'Hall-type' condition for the existence of such a coloring. We established this condition to be necessary if the independence system is a matroid, using the theory of matroid sums.

B. **Fractional coloring** [5, 9, 17, 35]

[35] is commented on above. [17] concerns fractional and list-multicoloring parameters, the fractional versions of which are mentioned in the title. The most interesting results there are that the fractional Hall-condition number, also called the Hall ratio, can be strictly less than the fractional chromatic number, and that the k-fold Hall number of the claw is asymptotically equal to $5k/3$. Incidentally, it is shown in [35] that the aforementioned Hall ratio is a lower bound on the full menagerie of fractional chromatic numbers definable with respect to probability measure spaces, as described in [35]. Papers in [5, 9] are surveys of recent developments in fractional graph theory, with special attention to fractional Euclidean Ramsey problems. Although results in this area, and especially on the matter of fractional list-coloring parameters [17, 35], seem a bit abstruse at present, this may be an area rich in applications to scheduling and allocation problems.

C. **Greedy coloring** [3]

It is proven that the Grundy number of the cube, Q_n, is $n+1$, unless $n=2$. In the process, a nice chunk of potentially useful game theory is exposed.

D. **Euclidean Ramsey problems** [5, 9, 11, 16, 20, 22]

Most of these problems are about coloring infinite graphs defined with reference to a set of distances. (Sample problem, dealt with in [22]: for which $\ell \geq 1$ and $s > 0$ can a $1 \times \ell$ planar rectangle be three-colored so that no two points a distance s apart are the same color?) Because of the Debruijn-Erdös compactness theorem, such problems are also about the finite subgraphs of these infinite graphs. These problems are at the gee-whiz stage now, but they may be fundamental in allocation or scheduling problems involving distance, and they certainly bring a new point of view to computational geometry.

III. Network Vulnerability [12, 18, 23, 24]

Except for [24], these papers are about a kind of network vulnerability defined by Johnson, inspired by unpublished work of Shields and Harary on a problem of the former. We start with a weighted network, a graph with weights on the vertices. An enemy wishes to dismantle this network by removing vertices until the sum of the weights on each connected component of what's left is less than some threshold value. The enemy pays for removing vertices – the cost for each vertex is a decreasing function of the weight at the vertex. (Decreasing, on the premise that the juiciest targets are hardest to defend.) The defenders of the network have to arrange the weights to exact the greatest payment for dismantling, from the enemy.

[18], which was actually written before [12], is a compendium of almost everything known about this situation up to the time of its writing. [24] is about a different sort of attack situation, thought up by Matthew Walsh: we still have a weighted network, and an enemy who wishes to dismantle it, as before, but now the enemy has no information about the structure of the network, and pays nothing for the destruction of a vertex. Rather, there is a *probability*, an increasing function of the weight at a vertex, that the vertex will be discovered and destroyed by the enemy. This time, the defenders wish to minimize the probability of dismantling.

IV. Structure of extremal graphs [6, 29, 34, 36, 37]

In earlier work it was shown that the graphs extremal for certain inequality are the "nearly strongly regular" graphs, regular graphs such that for some t, any two adjacent vertices have t common neighbors. In [6] it is shown that for such a graph of degree d on n vertices, if $t = 0$ and $2d = n-2$ (i.e., the joint neighborhood of any two adjacent vertices covers all but two vertices of the graph) then either the graph is a complete bipartite graph, or the graph is one of two non-bipartite graphs, of order 8 and 10 respectively. It is also shown that if $t > 0$ and $2d - t = n - 2$, then $n \leq 3t + 6$. In [29] it is shown that $n = 3t + 6$, with $2d - t = n - 2$, is achievable for every $t > 0$, by and only by the tripartite graph $K_{t+2,t+2,t+2}$ minus a two-factor consisting of K_3's.

[34] is about a number of inequalities relating the average degree of a graph to the average numbers of common neighbors and of joint neighborhood sizes; in every case, the extremal graphs are merely regular.

[36] and [37] are about (t, r)-regular graphs, non-cliques in which for every independent set of t vertices, the union of their neighbor sets has cardinality r. It turns out that for $t > 1$ and for $n \geq N(t, r)$, every such graph on n vertices must have a certain structure. In [36] a proof of this fact for $t = 2$ is given which shows that $r^2/16 < N(2, r) < r^2$, for $r > 3$, and in [37] a first proof of this fact, and a description of the structure and an estimate of $N(t, r)$, is given for every $t > 2$.

References

[1] D. G. Hoffman and P. D. Johnson Jr., Covering numbers and list colorings of $K_{m,n}$, Pan American Mathematical Journal 8 (1998), 13-22.

[2] M. Foust and C. A. Rodger, Hypergraph Designs, Festschrift for A. P. Street, JCMCC, 28 (1998), 125-140.

[3] D. G. Hoffman and P. D. Johnson Jr., Greedy colorings and the Grundy chromatic number of the n-cube, Bulletin of the ICA 26 (1999), 49-67.

[4] A. J. W. Hilton and P. D. Johnson Jr., The Hall number, the Hall index, and the total Hall number of a graph, Discrete Applied Math. 94 (1999), 227-245.

[5] P. D. Johnson Jr., More on the fractional chromatic number, Geombinatorics 9 (1999), 10-20.

[6] P. D. Johnson Jr. and K. J. Roblee, More extremal graphs for a maximal-joint-neighborhood, average-triangles-per-edge inequality, Congressus Numerantium 140 (1999), 87-95.

[7] M. Daven and C. A. Rodger, (k, g)-cages are 3-connected, Discrete Math., 199 (1999), 207-215.

[8] M. M. Cropper, J. L. Goldwasser, A. J. W. Hilton, D. G. Hoffman, and P. D. Johnson Jr., Extending the disjoint-representatives theorems of Hall, Halmos, and Vaughan to list-multicolorings of graphs, J. Graph Theory 33 (2000), 199-219.

[9] P. D. Johnson Jr., Infinite graphs and the fractional chromatic number: some recent surprises, Geombinatorics 9 (2000), 135-144.

[10] Ch. Eslahchi, A. J. W. Hilton, and P. D. Johnson Jr., Progress on the Hall number 2 problem, Australasian Journal of Combinatorics 21 (2000), 211-236.

[11] P. D. Johnson Jr., Problems posed in or arising from "Colorings of Metric Spaces", Geombinatorics 9 (2000), 170-179.

[12] J. Holliday and P. D. Johnson Jr., The Shields-Harary numbers of $K_n - e$, for continuous cost functions, Congressus Numerantium 147 (2000), 153-159.

[13] M. Daven and C. A. Rodger, The Johnson graph $J(v, k)$ has connectivity δ, Cong. Num., 139 (1999), 123-128.

[14] D. E. Bryant, S. El-Zanati, and C. A. Rodger, Maximal sets of hamilton cycles in $K_{n,n}$, J. Graph Theory, 33 (2000), 25-31.

[15] H. L. Fu and C. A. Rodger, Forest leaves and four-cycles, J. Graph Theory, 33 (2000), 161-166.

[16] A. Abrams and P. D. Johnson Jr., Yet another species of forbidden-distances chromatic number, Geombinatorics 10 (2001), 89-95.

[17] A. Daneshgar, A. J. W. Hilton, and P. D. Johnson Jr., Relations among the fractional chromatic, choice, Hall, and Hall-condition numbers of simple graphs, Discrete Math. 241 (2001), 189-199.

[18] Frank Harary and Peter Johnson, The Shields-Harary indices of vulnerability of a graph, Mathematical and Computer Modeling 34 (2001), 299-310.

[19] D. G. Hoffman, P. D. Johnson Jr., and A. D. Szlam, A new lemma in Ramsey theory, Journal of Combinatorial Mathematics and Combinatorial Computing 38 (2001), 123-128.

[20] P. D. Johnson Jr. and A. D. Szlam, A new connection between two kinds of Euclidean coloring problems, Geombinatorics 10 (2001), 172-178.

[21] P. D. Johnson Jr. and E. B. Wantland, Two problems concerning Hall's condition, Congressus Numerantium 152 (2001), 189-196.

[22] A. Bohannon, P. Johnson, J. Lanig, and R. Rubalcaba, The chromatic numbers of some Euclidean distance graphs, 10 (2001), 141-164.

[23] S. H. Holliday and P. D. Johnson Jr., On the constant-weight Shields-Harary numbers of a tree, Congressus Numerantium 153 (2001), 187-191.

[24] P. D. Johnson Jr. and M. Walsh, Another network vulnerability parameter, Congressus Numerantium 153 (2001), 25-32.

[25] E. J. Billington, H. L. Fu, and C. A. Rodger, Packing complete multipartite graphs with 4-cycles, J. Combinatorial Designs, 9 (2001), 107-127.

[26] J. McGee and C. A. Rodger, Path coverings with paths, J. Graph Theory, 36 (2001), 156-167.

[27] H. L. Fu and C. A. Rodger, Four-cycle systems with two-regular leaves, Graphs and Combinatorics, 17 (2001), 457-461.

[28] C. D. Leach and C. A. Rodger, Non-disconnecting disentanglements of amalgamated 2-factorizations of complete multipartite graphs, J. Combinatorial Designs 9 (2001), 460-467.

[29] P. D. Johnson Jr. and K. J. Roblee, On an extremal subfamily of an extremal family of nearly strongly regular graphs, Australasian Journal of Combinatorics 25 (2002), 279-284.

To appear

[30] P. D. Johnson Jr. and E. B. Wantland, Which graphs have Hall number 2? To appear in Ars Combinatoria.

[31] P. D. Johnson Jr., Hall's condition for list-coloring, and the Hall parameters; some recent developments. To appear in Discrete Math.

[32] C. D. Leach and C. A. Rodger, Fair hamilton decompositions of complete multipartite graphs, J. Combinatorial Theory (B), to appear.

[33] D. Bryant, Y. Chang, C. A. Rodger, and R. Wei, Two dimensional balanced sampling plans excluding contiguous units, Communications in Statistics, to appear.

Submitted

[34] D. G. Hoffman and P. D. Johnson Jr., Some inequalities for graphs derivable from Hölder's inequality. Submitted to Mathematical Inequalities and Applications.

[35] P. D. Johnson Jr. and C. A. Rodger, Coloring the vertices of a graph with measurable sets in a probability space; submitted to Discrete Math.

[36] P. D. Johnson Jr. and E. J. Morgan, Sharpening the Faudree-Knisley theorem on $(2, r)$-regularity. Submitted to the Bulletin of the ICA.

[37] R. E. Jamison and P. D. Johnson Jr., The structure of (t, r)-regular graphs of large order. Submitted to Discrete Math.

[38] A. J. W. Hilton, C. A. Rodger, and E. B. Wantland, Amalgamations of connected k-factorizations, submitted.

[39] M. Daven, J. A. MacDougall, and C. A. Rodger, Maximal sets of hamilton cycles in complete multipartite graphs, submitted.

[40] C. D. Leach and C. A. Rodger, Hamilton decompositions of complete multipartite graphs with any 2-factor leave, submitted.

[41] An efficient algorithm for multi-list coloring the two cliques graph.

[42] Multi-list colorings of matroids.